

A PROPERTY OF THE BIDIMENSIONAL SPHERE

MARIUS CAVACHI

ABSTRACT. It is natural to ask for a reasonable constant k having the property that any open set of area greater than k on a bidimensional sphere of area 1 always contains the vertices of a regular tetrahedron. We shall prove that it is sufficient to take $k = \frac{3}{4}$. In fact we shall prove a more general result. The interested reader will not have any problem in establishing that $\frac{3}{4}$ is the best constant with this property.

Keywords: area; open set; Haar measure; rotation group of the sphere.

Our result is the following:

Theorem 1. *Let n be a positive integer, and let S be a bidimensional sphere of area 1. If $M \subset S$ is an open set of area greater than $\frac{n-1}{n}$ and $X \subset S$ is a finite set with n elements, then there exists a rotation ρ of the sphere such that $\rho(X) \subset M$.*

In the proof, we use the following result whose proof we postpone:

Lemma 2. *Let $M, M' \subset S$ be open sets such that $\mathcal{A}(M) > \mathcal{A}(M')^1$. Then there exists a finite number of mutually disjoint spherical caps U_α and rotations ρ_α such that:*

- (i) $\bigcup_\alpha U_\alpha \subset M$;
- (ii) $M' \subset \bigcup_\alpha \rho_\alpha(U_\alpha)$;
- (iii) $M \setminus \bigcup_\alpha U_\alpha$ has non-empty interior.

Proof of the Theorem. Let μ be a Haar measure on $SO(3)$ such that $\mu(SO(3)) = 1$.

For any $A \subset S$, let Φ_A be the characteristic function of A .

Fix $a \in S$ and let $I_a^A \in \mathbb{R}$ be $I_a^A = \int_{SO(3)} \Phi_A \circ x(a) d\mu(x)$.

Remark 3. Note that if b is an arbitrary point on S , then $I_a^A = I_b^A$. Indeed if $\rho \in SO(3)$ is such that $\rho(a) = b$ (and such a ρ always exists), then:

$$\begin{aligned} I_b^A &= \int_{SO(3)} \Phi_A \circ x(\rho(a)) d\mu(x) = \int_{SO(3)} \Phi_A \circ (x \circ \rho)(a) d\mu(x \circ \rho) \\ &= \int_{SO(3)} \Phi_A \circ x(a) d\mu(x), \end{aligned}$$

since $d\mu(x \circ \rho) = d\mu(x)$, the Haar measure being rotation invariant.

¹For any $A \subset S$, $\mathcal{A}(A)$ denotes its area.

Moreover, if $B \subset S$ is an open set such that there exists $\rho_1 \in SO(3)$ with $\rho_1(A) = B$, then again $I_a^A = I_a^B$. Indeed,

$$\begin{aligned} I_a^B &= \int_{SO(3)} \Phi_{\rho(A)} \circ x(a) d\mu(x) = \int_{SO(3)} \Phi_A \circ \rho_1^{-1} \circ x(a) d\mu(x) \\ &= \int_{SO(3)} \Phi_A \circ (\rho_1^{-1} \circ x)(a) d\mu(\rho_1^{-1} \circ x) = I_a^A. \end{aligned}$$

Returning to the problem, if $X = \{a_1, \dots, a_n\}$, let

$$f : SO(3) \rightarrow \mathbb{R}, \quad f(x) = \sum_{i=1}^n \Phi_M \circ x(a_i).$$

Note that it is enough to find an $x \in SO(3)$ with $f(x) > n - 1$. Then, since $f(x)$ is an integer $\leq n$, we obtain $f(x) = n$ and hence $x(a_1), \dots, x(a_n) \in M$, which proves the Theorem. To find such an x , it is enough to show that

$$\int_{SO(3)} f(x) d\mu(x) > n - 1.$$

But this means that

$$\sum_{i=1}^n I_{a_i}^M > n - 1,$$

which is implied by

$$I_{a_i}^M > \frac{n-1}{n}$$

for each i , that is

$$I_a^M > \frac{n-1}{n}.$$

We divide the sphere S in n spherical lunes F_1, \dots, F_n of equal areas. Obviously, each F_i can be obtained as a rotation of F_1 . This implies:

$$1 = I_a^S = \sum_{i=1}^n I_a^{F_i} = n I_a^{F_1}, \quad \text{hence} \quad I_a^{F_1} = \frac{1}{n}.$$

Let now $M' = S \setminus F_n$. Then

$$I_a^{M'} = \sum_{i=1}^{n-1} I_a^{F_i} = \frac{n-1}{n}.$$

With U_α and ρ_α as in the Lemma, we deduce:

$$I_a^M > I_a^{\cup_\alpha U_\alpha} = \sum_\alpha I_a^{U_\alpha} = \sum_\alpha I_a^{\rho_\alpha(U_\alpha)} \geq I_a^{M'} = \frac{n-1}{n},$$

and the proof is complete. \square

Proof of the Lemma. Let $0 < m < 1$ and let C_i , for $i \in \{1, \dots, k\}$, be spherical caps of diameter d such that

$$\bigcup_{i=1}^k C_i = S,$$

and let P_i be the plane containing the center of S and parallel to the circle bounding C_i . If $\pi_i : S \rightarrow P_i$ is the orthogonal projection on P_i , we can choose d small enough such that:

- For any open $C \subset C_i$, we have $\mathcal{A}(\pi_i(C)) > m\mathcal{A}(C)$.
- For any $A \neq B \in C_i$, we have the inequality of segment lengths:

$$|\pi_i(A)\pi_i(B)| > m \cdot |AB|.$$

Define now $M_1 = C_1 \cap M$, $M_2 = C_2 \cap (M \setminus M_1)$, $M_3 = C_3 \cap (M \setminus M_1 \cup M_2)$, \dots , $M_k = C_k \cap (M \setminus M_1 \cup \dots \cup M_{k-1})$, and similarly construct M'_1, M'_2, \dots, M'_k .

Let $N_i = \pi_i(M_i)$, $N'_i = \pi_i(M'_i)$. For $1 - m$ close enough to 0, we have:

$$\sum_{i=1}^k \mathcal{A}(N_i) > \sum_{i=1}^k \mathcal{A}(N'_i).$$

In each plane P_i , we fix a side length ε square lattice. It can be proven (see [1, pag. 315,327]) that the number n_i of squares contained in N_i is

$$\frac{1}{\varepsilon^2} \mathcal{A}(N_i) + O\left(\frac{1}{\varepsilon}\right),$$

and analogously we have an approximation for the number n'_i of squares contained in N'_i . Hence, for small enough ε , we get

$$\sum_{i=1}^k n_i > \sum_{i=1}^k n'_i.$$

Therefore, we can choose an injection u from the set \mathcal{P}' of squares contained in $\bigcup_{i=1}^k N'_i$ into the set \mathcal{P} of squares contained in $\bigcup_{i=1}^k N_i$.

Let $P \in \mathcal{P}'$ (and hence $P \subset N'_i$ for some i), let $Q \in C_i$ be the point whose projection on P_i is the center of P , and let D_P be the spherical cap defined as the intersection of S with the ball centered in Q and of radius $\varepsilon/2$. Similarly, define $D_{u(P)}$, corresponding to $u(P)$. Clearly, $D_P = \rho_P(D_{u(P)})$ for some $\rho_P \in SO(3)$. We remove from M all the caps $D_{u(P)}$ and from M' all the caps D_P , for $P \in \mathcal{P}'$.

Define now $s = \mathcal{A}(M)$, $s' = \mathcal{A}(M')$. Since $\sum n'_i \varepsilon^2 \rightarrow \sum \mathcal{A}(N'_i)$, when $\varepsilon \rightarrow 0$, we can choose ε and $1 - m$ small enough such the above procedure removes from M and M' the sets \mathcal{M}_1 and \mathcal{M}'_1 of area greater than $\frac{1}{2}s'$.

Inductively, define S_i, S'_i as follows: $S_1 = M \setminus \mathcal{M}_1$ and $S'_1 = M' \setminus \mathcal{M}'_1$. By repeating the above process, obtain the sets S_2, S'_2 and so on.

Obviously, $\mathcal{A}(S'_t) < \left(\frac{1}{2}\right)^t \rightarrow 0$ as t grows to infinity. Since $\mathcal{A}(S_t) > s - s' > 0$, there exists some t such that

$$\mathcal{A}(S_t) > 4 \cdot \mathcal{A}(S'_t).$$

Once again, we go through the first step of the above construction applied to the sets S_t, S'_t with the difference that \mathcal{P}' will be the minimal set of all squares of lattices in P_i which cover $\bigcup_{i=1}^k N'_i$, and \mathcal{P} will contain all the squares of lattices with side length 2ε that are included in $\bigcup_{i=1}^k N_i$. Also, D_P will be the intersection of S with the ball centered at Q and of radius

$\frac{\varepsilon}{\sqrt{2}}$, and $D_{u(P)}$ is constructed analogously. The circle with the same center as $u(P)$, of radius $\frac{\varepsilon}{\sqrt{2}}$, is included in $u(P)$.

Letting the set of U_α be the set of all $D_{u(P)}$, the conditions (i) – (iii) in the Lemma are satisfied and the proof is complete. \square

REFERENCES

- [1] M. R. Murty, J. Esmonde, *Problems in algebraic number theory*, Springer-Verlag (2005)

”OVIDIUS” UNIVERSITY OF CONSTANȚA, 124 MAMAIA BVD 900527 CONSTANȚA, ROMANIA

E-mail address: mcavachi@yahoo.com